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Turbulence cascades across equilibrium spectra

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Recent work has suggested that the spectrum of fully developed turbulence is determined by the equilibrium statistics of the Euler equations. The problem is to reconcile this fact with the irreversible aspects of turbulence. The purpose of this paper is to show that a reconciliation is possible by producing simple systems whose spectrum can be deduced from an analysis at equilibrium, yet they exhibit an irreversible transfer of an appropriate variable from large scales to small scales and dissipation. The analysis includes the dependence of the spectrum on the rate of energy transfer. [S1063-651X(96)12609-X]

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Recent work [1] has suggested that the inertial (Kolmogorov) spectrum of turbulence can be deduced by considering Gibbsian equilibria of vortex filaments. This suggestion is orthogonal to the usual derivation of that spectrum [2], where the irreversible nature of the energy transfer between large and small sales constitutes the determining assumption. It is obvious that turbulence as a whole cannot be viewed as being in a Gibbsian equilibrium, since energy dissipation is one of its most salient features. The natural suggestion is that a full description of turbulence requires only a perturbation, small in some appropriate metric, of a Gibbsian equilibrium. This equilibrium must differ in substantial ways from the Hopf-Lee [3] equilibrium, with the difference due to the imposition of the constraints contained in the equations of fluid dynamics. Similar arguments have been made in two-dimensional turbulence.

It is not possible at present to decide whether these suggestions are valid by rigorous analysis of the equations of motion. The goal of the present paper is to show that these ideas are plausible by exhibiting simple systems whose spectrum can be determined from equilibrium considerations, yet, this spectrum is unchanged when these systems dissipate a quantity by first transporting it across that spectrum to the small scales, where it is lost. In these examples the amplitude of the spectrum is related to the rate of dissipation, as in Kolmogorov's law, not because the dissipation creates the spectrum, but, conversely, because the higher the amplitude of the spectrum, the more energy there is to dissipate. These models offer an interesting interpretation of the Kolmogorov theory, and open the door to the use of fluctuation-dissipation theorems and other near-equilibrium methods in turbulence.

The first example consists of a discretized baker's transformation [4] transporting a passive quantity through a ladder of scales. This example is linear, and is included because of its simplicity. The other example consists of a polymer folding in a solution, with mass transported from large scales to small scales and dissipated there. Various relations between turbulence theory and polymer dynamics have been presented over the years [1], and are closely linked to the relation between ϕ^4 field theories and vortex dynamics [5]. Our example exhibits noninteger exponents as in the Kolmogorov law, and is quite relevant to the broader issues of turbulence theory.

Consider first the unit square $D=\{x,y|0\leqslant x\leqslant 1,0\leqslant y\leqslant 1\}$; we are going to implement on D a discrete version of the baker's transformation $x'=2x-[2x]$, $y'=\frac{1}{2}(x+[2x])$, where $[x]$ denotes the largest integer $\leqslant x$. This transformation is ergodic and mixing [4], and thus is not an absurd paradigm for the growth in disorder typical of turbulence.

To discretize the transformation, divide D into boxes:

$$D_{ij}=\{x,y|ih\leqslant x\leqslant (i+1)h, \quad jh\leqslant y\leqslant (j+1)h\},$$

where $h=(1/n)$, $n=2^m$ for some integer m, and i,j take the values $0,1\ldots,(n-1)$. Furthermore, consider the discrete function $\phi=\{\phi_{ij}\}$, $\phi_{ij}=0$ or 1 on each D_{ij}. The discrete baker's transformation $(i,j)\rightarrow(i',j')$ is defined by

$$x=ih, \quad y=jh,$$

$$x'=2x-[2x], \quad y'=\frac{1}{2}(y+[2x]), \tag{1}$$

$$i'=x'/h, \quad j'=y'/h.$$

One can readily check that this transformation maps two pairs (i,j) on one (i',j') if i' is odd, and maps no (i,j), on (i',j') if i' is even. To construct the image ϕ' of ϕ under the transformation, consider for odd i' the sum s of the $\phi_{i,j}$ at the preimages of (i',j'), and set $\phi_{i',j'}=0$ if $s=0$, and $\phi_{i',j'}=1$ if $s\geqslant 1$. Then, if $s\leqslant 1$, set $\phi_{i'+1,j'}=0$; if $s=2$, set $\phi_{i'+1,j'}=1$. This construction preserves $\int\phi dx\,dy$ and (indeed, $\int\phi^q dx\,dy$ for all q). The use of an integer-valued function ϕ is designed to eliminate any entropy increase due to smoothing. It is easily seen that, given an initial ϕ with few "holes," the first few steps of the transformation (1) approximate well the first few steps of the continuous Baker's transformation. However, the sequence of discrete maps of ϕ is periodic with period $2m$. In Fig. 1 we exhibit initial data and every second subsequent state produced by this mapping, with $n=32$.

Start with initial data ϕ with few "holes;" for example, with a function ϕ which equals 1 in some $lh\times lh$ square, l integer, $l<n$ [as in Fig. 1(a)]. For the first few steps, the "energy" $\int\phi^2 dx\,dy$ moves to smaller and smaller scales, as in simple models of turbulent cascades. In the next m steps

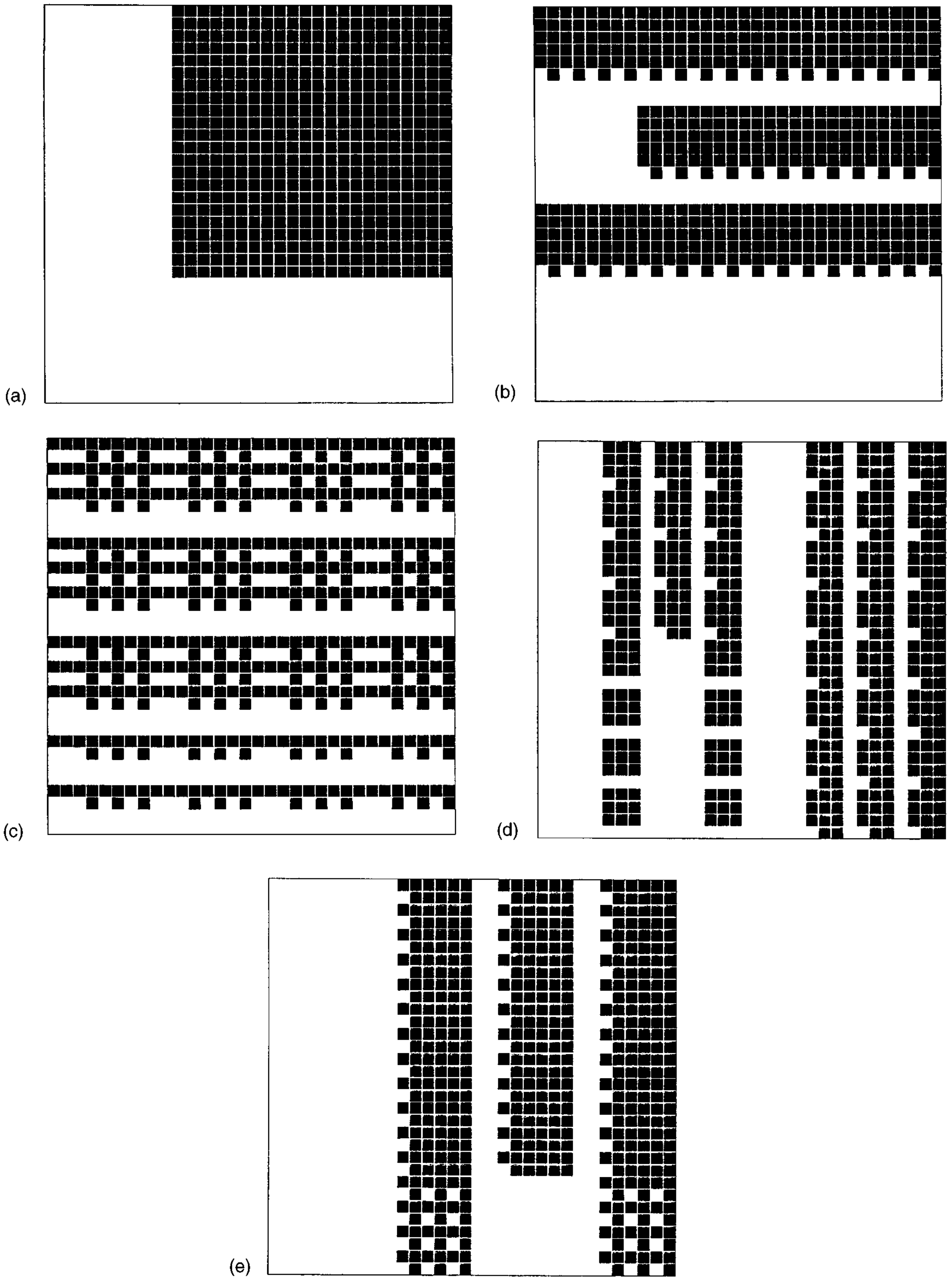

FIG. 1. (a)–(e) Successive states produced by the baker's transformation (every second state shown).

the energy comes back; the time average of the energy per scale is a constant independent of scale. One can readily find initial data for which the energy starts out roughly equidistributed among scales and remains so. We shall not bother to define a temperature and an entropy for this system, and shall identify the the equipartition spectrum with an equilibrium spectrum. Thus the energy spectrum has the form $E(k)=Ak^{\gamma}$, with $\gamma=0$, and its amplitude A depends on the amplitude of the initial data.

Now add an irreversible energy cascade to this model: Start with ''smooth'' initial data, for example $\phi_{ij}=1$ for $i,j\leq I, I=[\sqrt{zn}]$, where $z=\Sigma\phi_{ij}/n^2<1$ is the fraction of sites where $\phi_{ij}=1$, as in Fig. 1(a). Remove energy at small scales by setting ϕ_{ij} to 0 whenever $\phi_{ij}=1$ and $\phi_{i\pm 1,j}=0$, $\phi_{i,j\pm 1}=0$ (i.e., whenever there is an isolated 1). Feed energy at large scales by filling in the missing 1's whenever the initial data would have been recovered if it were not for the removal, i.e., once every $2m$ steps. Define a rate of energy

TABLE I. Estimates of the inverse spectral exponent μ in the presence of a cascade.

p	Polymer length N								
	600	900	1200	1500	1800	2100	2400	2700	3000
50	0.562	0.560	0.554	0.557	0.561	0.563	0.565	0.564	0.563
$N/6$	0.578	0.587	0.583	0.584	0.586	0.581	0.579	0.579	0.579
$N/2$	0.552	0.565	0.568	0.566	0.568	0.569	0.571	0.571	0.572
$2N/3$	0.586	0.586	0.584	0.581	0.570	0.576	0.576	0.576	0.574

dissipation ε as the ratio of the number of 1's removed per q steps divided by q (the limit $q\rightarrow\infty$ is reached in $2m$ steps). It is easy to see that the form of the spectrum is unchanged, and that for small z, $A=2m\varepsilon+o(z)$, i.e., $E(k)=2m\varepsilon k^{\gamma}$. One can also check that dissipation stops when $z>z_c$, $z_c=0.5625$, because for larger z no isolated $1's$ appear. For (z_c-z) small and positive, numerical experiment shows that ε is proportional to $(A-A_c)^2$, where A_c is the amplitude at $z=z_c$, i.e., $A=A_c+C\varepsilon^{0.5}$, where C is a constant. As advertised above, there is a relation between A and ε, not because the dissipation creates the spectrum but, conversely, because the more energy there is in the ''equilibrium'' scales, more of it can be dissipated. The recurrent behavior of the system has parallels in vortex dynamics [6].

We now turn to polymeric systems. The relation between polymer statistics and the physics of superfluid vortices is well known, as is the relation between polymers and classical vortex motion [1]. In particular, at a well-defined critical temperature the statistics of a polymer and of a vortex filament are identical. We consider mass transport across scales in a single polymer. At equilibrium, in which all the configurations of the polymer are equally likely, the relation between $\langle r_N\rangle$, the average end-to-end length of a polymer with N units (''monomers''), and the number of N of units is $\langle r_N\rangle\sim N^{\mu}$, where μ is the Flory exponent [7]. Flory's value for that exponent in three dimensions is $\mu=0.6$; the correct value is around $\mu=0.588$. One can define a mass density $\rho=\rho(\mathbf{x})$ associated with the polymer, where $\mathbf{x}$ is a spatial location, and a density correlation function for a dilute suspension of polymers $R(\mathbf{r})=\langle\rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r})\rangle$. An easy argument shows that for $\mathbf{r}$ small compared to the average length of a polymer, $R(\mathbf{r})=R(r)=\text{const}\times r^{D-3}$, where $r=|\mathbf{r}|$ is the length of $\mathbf{r}$ and $D=1/\mu\cong1.69$. Indeed, assuming that there is a monomer at $\mathbf{x}$ [something which happens with probability proportional to $\rho(\mathbf{x})$], the number of monomers in a sphere of radius r around $\mathbf{x}$ is proportional to r^D, the number of monomers in a shell between r, and $r+dr$ is proportional to r^{D-1}, and their density is proportional to r^{D-3}. A Fourier transform yields a ''mass spectrum'' $E(k)=Ak^{-D}$, where A is a constant and $E(k)$ is the Fourier transform of R integrated over a sphere of radius $k=|\mathbf{k}|$. The function $E(k)$ is a mass analog of the energy spectrum of hydrodynamics. The closeness of D to the real value of the Kolmogorov exponent is coincidental [8]. The constant in the ''mass spectrum'' law is proportional to the mass density; the constant of proportionality is dimensional and thus not worth pursuing.

We now endow this system with dynamics, and with a mass source at large scales and a mass sink at small scales. By way of dynamics we use the Madras-Sokal algorithm [9] that probes the equilibrium configurations of the polymer: First place the polymer on a cubic lattice, in particular so that the number of its configurations is finite for finite N; identify the bonds in the lattice with monomers. To allow for the finite volume of the monomers, make the lattice polymer self-avoiding, i.e., forbid any site from being visited twice. Then list all the isomorphisms of the lattice, i.e. all length-preserving transformations that map the lattice on itself; in three space dimensions, there are 48 such isomorphisms, represented by certain matrices with entries which are either 0, 1, or -1. Pick one end of the polymer to be the ''free end.'' The dynamics proceed as follows: Pick a bond on the polymer at random, rotate the part of the polymer between the bond picked and the free end by one of the isomorphisms picked at random with equal probabilities; if the result is self-avoiding, this is the next configuration of the polymer; if the result is not self-avoiding, the next configuration is taken as identical to the previous configuration.

This folding algorithm samples the equilibrium configurations of the polymer [9]. Following ample precedent (for example, Ref. [10]), we pretend that this algorithm represents the dynamics of the system, on the theory that true dynamics and the sampling algorithm both probe the same set of configurations with the same probabilities once the transients are gone.

We now modify this system so as to create a mass cascade across scales. Orient the polymer once and for all; after each rotation in the Madras-Sokal algorithm examine the polymer for ''hairpins,'' i.e. configurations in which the jth and $(j+2)$th segments (the numbering being sequential along the polymer) are antiparallel; the jth, $(j+1)$th, and $(j+2)$th then form a U-shaped structure (a ''hairpin;'' see Ref. [11]). This is the smallest-scale structure on a self-avoiding polymer. To create an energy sink at small scales, replace this hairpin by a single segment connecting the beginning of the jth segment to the end of the $(j+2)$th. This removes mass at the smallest scale; the result is obviously self-avoiding.

Every p steps (p to be chosen), add up the number M of segments lost, and add to one end of the polymer a straight line with M bonds. If it is not possible to construct such a line without perturbing the polymer, wait a few steps until it is possible. Experience shows that the wait is short. This addition of a straight line represents a mass injection at large scales. Mass is thus irreversibly transferred from large to small scales, where it is removed.

How does this mass transfer affect the spectrum? In Table I we present the result of a calculation of the exponent μ in the relation $\langle r_N\rangle\sim N^{\mu}$ deduced from numerical experiments with varying N and varying choices of p as a function of

N. The dependence of the results on the number of Monte Carlo steps is not shown, because the spread in the results provides a reliable check on the statistical error. The typical number of Monte Carlo moves to equilibration was $K = 40N$, with only the last half used in the averaging, the rest being used to forget the initial conditions. We conclude that the value of μ in the presence of the cascade is $\mu = 0.57$ with a standard deviation of 0.005, independently of p. The corresponding exponent in the spectral power law $E(k) = Ak^{-D}$ is $D = \mu = 1.75$, with a standard deviation of 0.01. Note that these results are not sharp enough to exclude the possibility that there is a small difference between the equilibrium value of D and the value of D in the presence of a cascade, but if there is one, it is small (comparable with 0.01) and not at all comparable to the difference of 11/3 between the Hopf equilibrium exponent and the Kolmogorov exponent; the latter large difference is one reason for the belief that the inertial range of turbulence is far from equilibrium. (Note that a rough calculation of the same general kind had already been carried out for a vortex filament model in Ref. [12]).

A rate of mass transfer ε can now be estimated: The number of attempted foldings per unit length of the polymer is proportional to the number of Monte Carlo steps divided by the number N of monomers. (The relevant number is the number of folding per unit length rather than the number of foldings by itself because we are interested in the rate of energy transfer locally in space, in analogy with hydrodynamics; without the division by N, the longer the polymer the less each section would be folded per unit time). The rate of energy transfer ε is the limit of the limit of number of segments removed in K steps, divided by KN, as K increases.

Numerical calculation shows that ε is, within the statistical error, proportional to the number of monomers N; this is heuristically transparent, as we expect the number of ''hairpins'' formed in a given number of steps to be proportional to the number of monomers. The spectral law is thus $E(k) = C\varepsilon k^{-D}$, where C is a dimensional constant. As in the baker's transformation, the appearance of ε in the law does not mean that the mass cascade creates the spectrum, but only that the more mass there is, the larger the energy transfer.

One may wonder whether the fact that ε appears with exponent 1 in both examples has some significance which detracts from the value of our examples. This is not the case. The linear relation between E and ε in the first example is only an approximation that holds at small amplitude. In the second example, the linear relation is a consequence of the proportionality between the number of hairpins and the length of the polymer in three space dimensions. One can expect the number of hairpins to grow faster than that length in the recurring two-dimensional case [13], and thus yield an exponent less than 1 for ε. The same expectation holds in vortex dynamics near critical points [1].

In summary, we have produced simple examples that demonstrate the possibility of having spectra of Kolmogorov type be determined by equilibrium considerations, and remain unmoved by the presence of energy transfer across scales. Thus there is not necessarily a dichotomy between the kind of methods used in standard ''equilibrium'' field theories and the kind of methods appropriate for turbulence.

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[1] A. J. Chorin, *Vorticity and Turbulence* (Springer, New York, 1994); A. J. Chorin, Lect. Appl. Math. **31**, 235 (1996).

[2] W. D. McComb, *Physical Theories of Turbulence* (Cambridge University Press, Cambridge, 1989); M. Lesieur, *Turbulence in Fluids* (Kluwer, Dordrecht, 1990).

[3] E. Hopf, J. Rat. Mech. Anal. **1**, 87 (1952); T. D. Lee, Q. Appl. Math. **10**, 69 (1952).

[4] J. Moser, *Stable and Random Motion in Dynamical Systems* (Princeton University Press, Princeton, 1973).

[5] H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific, Singapore, 1989).

[6] D. Samuel and R. Donnelly, Phys. Rev. Lett. **64**, 1385 (1990).

[7] P. D. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell University Press, Ithaca, NY, 1971).

[8] J. Akao (unpublished).

[9] N. Madras and A. Sokal, J. Stat. Phys. **50**, 109 (1988).

[10] Ph. Martin, J. Stat. Phys. **16**, 149 (1977).

[11] A. J. Chorin, Commun. Pure Appl. Math. **39**, S47 (1986).

[12] A. J. Chorin and J. Akao, Physica D **52**, 403 (1991).

[13] G. Lawler, *Intersections of Random Walks* (Birkhauser, Boston, 1991).